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A quasiseparable approach to five-diagonal CMV and Fiedler matrices

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ABSTRACT

Recent work in the characterization of structured matrices in terms of characteristic polynomials of principal submatrices is furthered in this paper. Some classical classes of matrices with quasiseparable structure include tridiagonal (related to real orthogonal polynomials) and banded matrices, unitary Hessenberg matrices (related to Szegö polynomials), and semiseparable matrices, as well as others. Hence working with the class of quasiseparable matrices provides new results which generalize and unify classical results.
Previous work has focused on characterizing $(H, 1)$-quasiseparable matrices, matrices with order-one quasiseparable structure that are also upper Hessenberg. In this paper, the authors introduce the concept of a twist transformation, and use such transformations to explain the relationship between $(H, 1)$-quasiseparable matrices and the subclass of $(1, 1)$-quasiseparable matrices (without the upper Hessenberg restriction) which are related to the same systems of polynomials. These results generalize the discoveries of Cantero, Fiedler, Kimura, Moral and Velázquez of five-diagonal matrices related to Horner and Szegö polynomials in the context of quasiseparable matrices.

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1. Introduction

Various polynomial systems $\{r_k(x)\}_{k=0}^n$ are often associated with Hessenberg matrices $H = [m_{ij}]_{ij=1}^n$ as scaled characteristic polynomials of their principal submatrices; that is, via the relation

$$r_0(x) = \lambda_0, \quad r_k(x) = \lambda_0\lambda_1 \cdots \lambda_k \det(xI - H_{k\times k}), \quad k = 1, \ldots, n. \tag{1.1}$$

Moreover, the relation (1.1) in concert with the results of [1] allows the establishment of a bijection if $\lambda_k = \frac{1}{m_{k+1,k}}$ and $\lambda_0,\ \lambda_n$ are two parameters, so

$$\{r_k(x)\}_{k=0}^n \longleftrightarrow \{H, \lambda_0, \lambda_n\}. \tag{1.2}$$

1.1. From Hessenberg to five-diagonal matrices. Two examples

It is widely known that Szegö polynomials $\{\phi_k^{\#}(x)\}_{k=0}^n$ orthogonal on the unit circle are connected via (1.1) with a certain (almost[1]) unitary Hessenberg matrix

$$M = \begin{bmatrix} -\rho_0^*\rho_1 & -\rho_0^*\mu_1\rho_2 & -\rho_0^*\mu_1\mu_2\rho_3 & \cdots & -\rho_0^*\mu_1\mu_2\mu_3\cdots\mu_{n-1}\rho_n \\ \mu_1 & -\rho_1^*\rho_2 & -\rho_1^*\mu_2\rho_3 & \cdots & -\rho_1^*\mu_2\mu_3\cdots\mu_{n-1}\rho_n \\ 0 & \mu_2 & -\rho_2^*\rho_3 & \cdots & -\rho_2^*\mu_3\cdots\mu_{n-1}\rho_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \mu_{n-1} & -\rho_{n-1}^*\rho_n \end{bmatrix}, \tag{1.3}$$

where ρ_k are *reflection coefficients*[2] and μ_k are *complementary parameters*. The details on this relation can be found in [4–12]. However, the bijection (1.2) implies that for a given system of Szegö polynomials, the only Hessenberg matrix related to that system via (1.1) is M. The situation is much different if we do not restrict the matrix to the class of strictly upper Hessenberg matrices.

It was found by Kimura [13] and later independently by Cantero et al. [14,15,16] (see also [17,9]) that Szegö polynomials are also related via (1.1) (with $\lambda_k = \frac{1}{\mu_k}$) to the five-diagonal matrix

$$\mathcal{K} = \begin{bmatrix} -\rho_0^*\rho_1 & \rho_0^*\mu_1 & 0 & & & & & \\ -\mu_1\rho_2 & -\rho_1^*\rho_2 & -\mu_2\rho_3 & \mu_2\mu_3 & & & & \\ \mu_1\mu_2 & \rho_1^*\mu_2 & -\rho_2^*\rho_3 & \rho_2^*\mu_3 & 0 & & & \\ & 0 & -\mu_3\rho_4 & -\rho_3^*\rho_4 & -\mu_4\rho_5 & \mu_4\mu_5 & & \\ & & \mu_3\mu_4 & \rho_3^*\mu_4 & -\rho_4^*\rho_5 & \rho_4^*\mu_5 & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{1.4}$$

which has been called a CMV matrix since the paper [14] triggered deep interest in the orthogonal polynomials community, see, for instance, a nice survey [18]. It is often reputed that CMV matrices might be better than unitary Hessenberg matrices for studying properties of polynomials orthogonal on the unit circle (mostly because of their banded structure).

Along with the discovery of CMV matrices, other non-Hessenberg matrices related to important systems of polynomials via (1.1) were discovered. Consider the well known companion matrix

$$C = \begin{bmatrix} -m_1 & -m_2 & \cdots & -m_{n-1} & -m_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{1.5}$$

As above, the bijection (1.2) implies that for the system of polynomials related via (1.1), C is the only such related matrix that is also upper Hessenberg. Omitting this Hessenberg restriction, however, again permits other related matrices. It was shown by Fiedler [19] that the five-diagonal matrix

[1] Throughout the paper, matrices referred to as unitary Hessenberg are almost unitary, differing from unitary in the last column. Specifically, $M = UD$ for a unitary matrix U and diagonal matrix $D = \text{diag}\{1, \ldots, 1, \rho_n\}$.

[2] Reflection coefficients are also known in various contexts as Schur parameters [2], Verblunsky coefficients [3].

$$F = \begin{bmatrix} -m_1 & -m_2 & 1 & & & & & \\ 1 & 0 & 0 & 0 & & & & \\ 0 & -m_3 & 0 & -m_4 & 1 & & & \\ & 1 & 0 & 0 & 0 & 0 & & \\ & & 0 & -m_5 & 0 & -m_6 & 1 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{1.6}$$

is also related to the same set of polynomials via (1.1).

1.2. Quasiseparable approach. Twist transformation

Both matrices M and C of the previous section are special cases of *quasiseparable* matrices, to be defined next. This property is significant, as although the CMV matrices $\mathcal{K}$ and Fiedler matrices F of the previous section do not have the Hessenberg property of M and C, they do preserve their quasiseparable property.

Definition 1.1 (*Rank definition of* $(1, 1)$*-quasiseparable matrices*)**.** A matrix A is called $(1, 1)$*-quasiseparable* (i.e., *order one quasiseparable*) if

$$\max_{2 \leqslant i \leqslant n-2} \operatorname{rank} A(1 : i, i+1 : n) = \max_{2 \leqslant i \leqslant n-2} \operatorname{rank} A(i+1 : n, 1 : i) = 1.$$

It is easy to see that both CMV matrices and Fielder matrices are $(1, 1)$-quasiseparable. Indeed, every submatrix $A(1 : i, i+1 : n)$ or $A(i+1 : n, 1 : i)$ of both matrices consists of at most two nonzero elements in the same row or column and, therefore, $\operatorname{rank} A(1 : i, i+1 : n) = \operatorname{rank} A(i+1 : n, 1 : i) = 1$ for all $i = 2, \dots n-2$. We refer to [20] for the details of the proof that both unitary Hessenberg matrices (1.3) and companion matrices (1.5) are also $(1, 1)$-quasiseparable.

It is well-known, see e.g. [21], that an equivalent definition of $(1, 1)$-quasiseparable matrices can be given in terms of the small number of parameters they are described by. Such sparse representations are often at the heart of fast algorithms involving this and similar classes of structured matrices.

Definition 1.2 (*Generator definition of* $(1, 1)$*-qs matrices*)**.** A matrix A is called $(1, 1)$-quasiseparable if it can be represent in the form

$$\begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & \cdots & \cdots & g_1b_2\cdots b_{n-1}h_n \\ p_2q_1 & d_2 & g_2h_3 & \cdots & \cdots & g_2b_3\cdots b_{n-1}h_n \\ p_3a_2q_1 & p_3q_2 & d_3 & \cdots & \cdots & g_3b_4\cdots b_{n-1}h_n \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & d_{n-1} & g_{n-1}h_n \\ p_na_{n-1}\cdots a_2q_1 & p_na_{n-1}\cdots a_3q_2 & p_na_{n-1}\cdots a_4q_3 & \cdots & p_nq_{n-1} & d_n \end{bmatrix},$$

where the parameters $\{q_k, a_k, p_k, d_k, g_k, b_k, h_k\}$, all scalars, are called *generators* of A.

Remark 1.3. The choice of generators of a $(1, 1)$-quasiseparable matrix is not unique.

One of many useful properties of $(1, 1)$-quasiseparable matrices is the existence of two-term[3] recurrence relations in terms of its generators for polynomials related to them via (1.1).

Theorem 1.4 [22]**.** *Let* $\{r_k(x)\}_{k=0}^{n}$ *be a system of polynomials related to a* $(1, 1)$*-quasiseparable matrix* A *via* (1.1)*. Then they satisfy two-term recurrence relations*

[3] To distinguish from the (different) Szegö-type recurrence relations (specified later in (3.5)), (1.7) are referred to as EGO-type recurrence relations.

Table 1
Generators of unitary Hessenberg matrix.

d_k	a_k	b_k	q_k	g_k	p_k	h_k
$-\rho_{k-1}^*\rho_k$	0	μ_k	μ_k	$-\rho_{k-1}^*\mu_k$	1	ρ_k

Table 2
Generators of CMV matrix.

k	d_k	a_k	b_k	q_k	g_k	p_k	h_k
Odd	$-\rho_{k-1}^*\rho_k$	0	μ_k	μ_k	$-\rho_{k-1}^*\mu_k$	1	ρ_k
Even	$-\rho_{k-1}^*\rho_k$	μ_k	0	$-\rho_{k-1}^*\mu_k$	μ_k	ρ_k	1

$$\begin{bmatrix} F_0(x) \\ r_0(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda_0 \end{bmatrix}, \quad \begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \lambda_k \begin{bmatrix} a_k b_k x - c_k & -q_k g_k \\ p_k h_k & x - d_k \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}, \tag{1.7}$$

where $c_k = d_k a_k b_k - q_k p_k b_k - g_k h_k a_k$.

What one can get immediately from this theorem is that the interchange of lower and upper generators as in

$$a_k \longleftrightarrow b_k, \quad p_k \longleftrightarrow h_k, \quad q_k \longleftrightarrow g_k \tag{1.8}$$

for some k does not change the recurrence relations (1.7) and, hence, does not change polynomials $\{r_k(x)\}_{k=0}^n$. We propose to call an operation consisting of interchanging the pairs of generators in (1.8) for some k a *twist transformation*.

We next show that both CMV and Fiedler matrices can be obtained via twist transformations from unitary Hessenberg and companion matrices, respectively.

Example 1.5 (*Unitary Hessenberg and CMV matrices*)**.** By comparing a set of (1, 1)-quasiseparable generators of the unitary Hessenberg matrix (Table 1) and CMV matrix (Table 2), we conclude that the second is obtained from the first via twist transformations for even indices.

Example 1.6 (*Companion and Fiedler matrices*)**.** Similarly, comparing Tables 3 and 4, one can see that the Fiedler matrix is obtained from the companion matrix via twist transformations for odd indices $k > 1$.

The invariance of systems of polynomials under twist transformation together with Examples 1.5 and 1.6 explains why unitary Hessenberg and CMV matrices as well as companion and Fiedler matrices share the same systems of characteristic polynomials.

1.3. Main results and structure of the paper

As we have mentioned already, unitary Hessenberg and companion matrices are both strictly upper Hessenberg and (1, 1)-quasiseparable. Such matrices have often been called $(H, 1)$-quasiseparable (see Definition 2.5). In Section 2 of the present paper we study matrices obtained from $(H, 1)$-quasiseparable matrices via twist transformation (which we call *twisted* $(H, 1)$-quasiseparable matrices). We also show that every $(H, 1)$-quasiseparable matrix can be transformed into a five-diagonal matrix via twist transformation. That is, the set of twisted $(H, 1)$-quasiseparable matrices for a given $(H, 1)$-quasiseparable matrix always contains a five-diagonal matrix.

The next part of the paper is devoted to the study of recurrence relations for (scaled) characteristic polynomials of principal submatrices of five-diagonal twisted $(H, 1)$-quasiseparable matrices. In the recent paper [20], the authors derived specific recurrence relations for various subclasses of

Table 3
Generators of companion matrix.

k	d_k	a_k	b_k	q_k	g_k	p_k	h_k
$k = 1$	$-m_1$	–	–	1	1	–	–
$k \neq 1$	0	0	1	1	0	1	$-m_k$

Table 4
Generators of Fiedler matrix.

k	d_k	a_k	b_k	q_k	g_k	p_k	h_k
$k = 1$	$-m_1$	–	–	1	1	–	–
$k > 1$ Odd	0	1	0	0	1	$-m_k$	1
Even	0	0	1	1	0	1	$-m_k$

$(H, 1)$-quasiseparable matrices. Moreover, because of the bijection (1.2), they have obtained a full characterization of subclasses of $(H, 1)$-quasiseparable matrices via recurrence relations satisfied by polynomials related to them via (1.1). We give a brief survey of the results of [20] in Section 3 in order to exploit them in Section 4 in connection with five-diagonal matrices.

In particular, we derive the class of five-diagonal matrices which are connected to polynomials satisfying three-term recurrence relations

$$\begin{aligned} &r_0(x) = \alpha_0, \quad r_1(x) = (\alpha_1 x + \beta_1) \cdot r_0(x), \\ &r_k(x) = (\alpha_k x + \beta_k) \cdot r_{k-1}(x) + (\gamma_k x + \delta_k) \cdot r_{k-2}(x), \quad \alpha_k \neq 0, \quad k = 2, \ldots, n. \end{aligned} \tag{1.9}$$

It was shown by Geronimus [23] that, under the additional restriction of $\rho_k \neq 0$ for every k, the corresponding Szegö polynomials $\{\phi_k^\#(x)\}_{k=0}^n$ satisfy three-term recurrence relations

$$\begin{aligned} &\phi_0^\#(x) = \frac{1}{\mu_0}, \quad \phi_1^\#(x) = \frac{1}{\mu_1}(x \cdot \phi_0^\#(x) + \rho_1 \rho_0^* \cdot \phi_0^\#(x)), \\ &\phi_k^\#(x) = \left[\frac{1}{\mu_k} \cdot x + \frac{\rho_k}{\rho_{k-1}} \frac{1}{\mu_k}\right] \phi_{k-1}^\#(x) - \frac{\rho_k}{\rho_{k-1}} \frac{\mu_{k-1}}{\mu_k} \cdot x \cdot \phi_{k-2}^\#(x), \end{aligned} \tag{1.10}$$

which are of type (1.9). One can also check that the Horner polynomials, related to the Fiedler matrix (1.6) via (1.1), satisfy the recurrence relations

$$p_k(x) = \left(x + \frac{m_k}{m_{k-1}}\right) p_{k-1}(x) - \frac{m_k}{m_{k-1}} x \cdot p_{k-2} \tag{1.11}$$

under the restriction $m_k \neq 0$ for every k. Recurrence relations (1.11) are of type (1.9) as well. Hence, both five diagonal CMV matrices (1.4) and Fiedler matrices (1.6) belong to the new class of five-diagonal matrices related to polynomials satisfying three-term recurrence relations via (1.1). This result as well as other results on polynomials related to five-diagonal matrices via (1.1) are presented in Section 4.

Also presented in Section 4 are conditions on the entries of five-diagonal matrices in order to be related via (1.1) to polynomials satisfying three-term (1.9) and two-term recurrence relations (4.16). Specifically, we show that given a five-diagonal matrix having the following sign patterns (where highlighted entries represent nonzero entries),

$$\begin{bmatrix} \star & \star & 0 & & & & & \\ \boxed{\star} & \star & \boxed{\star} & \star & & & & \\ \star & \star & \star & \star & 0 & & & \\ & 0 & \boxed{\star} & \star & \boxed{\star} & \star & & \\ & & \star & \star & \star & \star & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \quad \begin{bmatrix} \star & \star & 0 & & & & & \\ \star & \star & \star & \boxed{\star} & & & & \\ \boxed{\star} & \star & \star & \star & 0 & & & \\ & 0 & \star & \star & \star & \boxed{\star} & & \\ & & \boxed{\star} & \star & \star & \star & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

there exist three-term (1.9) and two-term recurrence relations (4.16), respectively, for the corresponding system of polynomials. Conversely, given such recurrence relations, then among the five-diagonal matrices corresponding to those recurrence relations, it is possible to choose one of the corresponding sign pattern. Interested reader can find more on the connection between five-diagonal matrices and polynomial systems in the sequel paper [24].

Finally, in Section 5 we derive a nested decomposition of twisted $(H, 1)$-quasiseparable matrices which can also be used to obtain them from the original $(H, 1)$-quasiseparable matrices.

2. Twist transformations and twisted $(H, 1)$-quasiseparable matrices

2.1. Twist transformations

A system of polynomials can be related to many distinct $(1, 1)$-quasiseparable matrices (Definition 1.2) via (1.1). For instance, a nonsymmetric $(1, 1)$-quasiseparable matrix and its transpose share the same system of polynomials. In this section we show how, given a $(1, 1)$-quasiseparable matrix, one can obtain another $(1, 1)$-quasiseparable matrix related to the same system of polynomials as the original one via (1.1).

Definition 2.1 (*Twist transformation*)**.** An $n \times n$ $(1, 1)$-quasiseparable matrix $\widetilde{A}$ having generators $\{\tilde{p}_k, \tilde{q}_k, \tilde{a}_k, \tilde{g}_k, \tilde{h}_k, \tilde{b}_k, \tilde{d}_k\}$ is obtained via twist transformation from another $n \times n$ $(1, 1)$-quasiseparable matrix A with generators $\{p_k, q_k, a_k, g_k, h_k, b_k, d_k\}$ if there exists a set $K \subset \{1, 2, \ldots, n\}$ such that

$$\begin{cases} \tilde{q}_1 = g_1, \quad \tilde{g}_1 = q_1, \quad \tilde{d}_1 = d_1 & \text{if } 1 \in K, \\ \tilde{p}_k = h_k, \quad \tilde{q}_k = g_k, \quad \tilde{a}_k = b_k, & \\ \tilde{h}_k = p_k, \quad \tilde{g}_k = q_k, \quad \tilde{b}_k = a_k, \quad \tilde{d}_k = d_k & \text{if } k \in K, \quad 1 < k < n, \\ \tilde{p}_n = h_n, \quad \tilde{h}_n = p_n, \quad \tilde{d}_n = d_n & \text{if } n \in K, \end{cases} \tag{2.1}$$

and all other generators of $\widetilde{A}$ and A are equal. Additionally, if K contains a single index, we call the transformation an elementary twist transformation.

In other words, $\widetilde{A}$ is obtained from A via the interchange of pairs of generators as in

$$a_k \longleftrightarrow b_k, \quad p_k \longleftrightarrow h_k, \quad q_k \longleftrightarrow g_k$$

for some subset of indices k. The fact that each pair consists of one generator from the upper part of the matrix and one from the lower part of the matrix (see Definition 1.2) is why we propose to call the operations of (2.1) *twist transformations*.

The significant feature of the twist transformation is that it transforms one $(1, 1)$-quasiseparable matrix into another while preserving the coefficients of the recurrence relations (1.7) and, thus, also preserving the characteristic polynomials of all of their submatrices. The next theorem exploits this fact.

Theorem 2.2. *The system of polynomials related to a* $(1, 1)$*-quasiseparable matrix A via* (1.1) *is invariant under twist transformations.*

Proof. It suffices to prove the theorem for an elementary twist transformation with $K = \{k\}$. Let $\widetilde{A}$ be the matrix obtained from A via (2.1) and $\{r_k(x)\}_{k=0}^n$ and $\{\tilde{r}_k(x)\}_{k=0}^n$ be the system of polynomials related to A and $\widetilde{A}$ via (1.1), respectively. Considering the recurrence relations (1.7) for polynomials related to $(1, 1)$-quasiseparable matrices via (1.1) and noticing that

$$\begin{aligned} &\tilde{a}_k\tilde{b}_k = a_kb_k, \quad \tilde{p}_k\tilde{h}_k = p_kh_k, \quad \tilde{d}_k = d_k, \\ &\tilde{d}_k\tilde{a}_k\tilde{b}_k - \tilde{q}_k\tilde{p}_k\tilde{b}_k - \tilde{g}_k\tilde{h}_k\tilde{a}_k = d_ka_kb_k - q_kp_kb_k - g_kh_ka_k. \end{aligned}$$

We conclude that both systems of polynomials $\{r_k(x)\}_{k=0}^n$ and $\{\tilde{r}_k(x)\}_{k=0}^n$ satisfy the same recurrence relations and, hence, coincide. $\square$

Corollary 2.3. *Examples 1.5 and 1.6 show that CMV matrices (1.4) and Fiedler matrices (1.6) are obtained via twist transformations from unitary Hessenberg matrices (1.3) and companion matrices (1.5), respectively. Hence, unitary Hessenberg matrices and CMV matrices share the same systems of characteristic polynomials, as do companion matrices and Fiedler matrices.*

Corollary 2.4. *For an arbitrary $(1,1)$-quasiseparable matrix A of size n, there are at least 2^n (possibly not distinct) matrices obtained from A via twist transformations related to the same system of polynomials as A via (1.1).*

2.2. Twisted $(H,1)$-quasiseparable matrices

Following [20,25], we define the class of matrices which are both strictly[4] upper Hessenberg and $(1,1)$-quasiseparable. The definition, like Definition 1.2, is given in terms of generators, see [20,25] for an equivalent definition in terms of ranks.

Definition 2.5 (*Generator definition of $(H,1)$-quasiseparable matrices*)**.** A matrix A is called $(H,1)$-quasiseparable if it can be represented in the form

$$A = \begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & \cdots & \cdots & g_1b_2\ldots b_{n-1}h_n \\ q_1 & d_2 & g_2h_3 & \cdots & \cdots & g_2b_3\ldots b_{n-1}h_n \\ 0 & q_2 & d_3 & \cdots & \cdots & g_3b_4\ldots b_{n-1}h_n \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & q_{n-2} & d_{n-1} & g_{n-1}h_n \\ 0 & \cdots & \cdots & 0 & q_{n-1} & d_n \end{bmatrix}, \tag{2.2}$$

where the parameters $\{q_k \neq 0, d_k, g_k, b_k, h_k\}$ are called *generators* of A.

Remark 2.6. Comparing Definitions 1.2 and 2.5 one can easily see that a $(1,1)$-quasiseparable matrix is $(H,1)$-quasiseparable if and only if it has a choice of generators such that $a_k = 0, p_k = 1, q_k \neq 0$.

It is easy to check that both unitary Hessenberg matrices (1.3) and companion matrices (1.5) are $(H,1)$-quasiseparable (in fact, the generators listed in Tables 1 and 3 demonstrate this fact). As we have seen, CMV matrices (1.4) and Fiedler matrices (1.6) can be obtained from these matrices via twist transformations. In order to generalize these results, we define next the entire class of matrices which can be obtained from $(H,1)$-quasiseparable matrices via twist transformations.

Definition 2.7 (*Twisted $(H,1)$-quasiseparable matrices*)**.** A $(1,1)$-quasiseparable matrix A is called twisted $(H,1)$-quasiseparable if it can be obtained from an $(H,1)$-quasiseparable matrix via twist transformations.

Applying twist transformations to the matrix (2.2) explicitly yields the following alternative definition of twisted $(H,1)$-quasiseparable matrices in terms of their generators.

Definition 2.8 (*Generator definition of twisted $(H,1)$-quasiseparable matrices*)**.** A $(1,1)$-quasiseparable matrix A is twisted $(H,1)$-quasiseparable if and only if it has a choice of generators $\{p_k, q_k, a_k, g_k, h_k, b_k, d_k\}$ such that

$$\begin{cases} q_1 \neq 0 \quad \text{or} \quad g_1 \neq 0, \\ a_k = 0, \quad q_k \neq 0, \quad p_k = 1 \quad \text{or} \quad b_k = 0, \quad g_k \neq 0, \quad h_k = 1, \qquad k = 2\ldots n-1, \\ p_n = 1 \quad \text{or} \quad h_n = 1. \end{cases}$$

[4] That is having nonzero subdiagonal elements.

It is clear that a twist transformation is completely determined by the indices $k \in K \subset \{1, 2, \ldots, n\}$ specifying the pairs to be interchanged, as in Definition 2.1. Such is called the *pattern* of the twist transformation, and is used next to distinguish between twisted $(H, 1)$-quasiseparable matrices corresponding to the same $(H, 1)$-quasiseparable matrix by various twist transformation.

Definition 2.9 (*Pattern of twisted $(H, 1)$-quasiseparable matrices*)**.** For an arbitrary, $n \times n$, twisted $(H, 1)$-quasiseparable matrix A, a *pattern* of A is a subset $K \subset \{1, 2, \ldots, n\}$ that defines a twist transformation that takes A to some $(H, 1)$-quasiseparable matrix. Equivalently, K is a pattern of A if there exist generators of A satisfying

$$\begin{cases} q_1 \neq 0 & & & \text{if } 1 \notin K, \\ g_1 \neq 0 & & & \text{if } 1 \in K, \\ a_k = 0, & q_k \neq 0, & p_k = 1 & \text{if } k \notin K,\ 1 < k < n, \\ b_k = 0, & g_k \neq 0, & h_k = 1 & \text{if } k \in K,\ 1 < k < n, \\ p_n = 1 & & & \text{if } n \notin K, \\ h_n = 1 & & & \text{if } n \in K. \end{cases} \tag{2.3}$$

We will also use the notation of a sequence of binary digits $(i_1, i_2, \ldots, i_n)$ to represent a pattern, where $i_k = 1$ when $k \in K$ and $i_k = 0$ when $k \notin K$. With this notation, we write

$$A = H(i_1, i_2, \ldots, i_n)$$

to mean that A is brought to $(H, 1)$-quasiseparable form by the twist transformation having pattern $(i_1, i_2, \ldots, i_n)$.

Example 2.10. According to this definition, any $(H, 1)$-quasiseparable matrix H of size n is $H(\underbrace{0, 0, \ldots, 0}_{n})$ and its transpose is $H(\underbrace{1, 1, \ldots, 1}_{n})$.

Example 2.11. Comparing the generators of unitary Hessenberg matrices (Table 1) and CMV matrices (Table 2), it is easy to see that CMV matrices have pattern $(0, 1, 0, 1, \ldots)$. A similar observation shows that Fiedler matrices have pattern $(1, 0, 1, 0, 1, \ldots)$.

The previous example restates the observations of Examples 1.5 and 1.6 in the terminology of patterns and twisted $(H, 1)$-quasiseparable matrices. It also suggests a basic connection between $(H, 1)$-quasiseparable matrices and five-diagonal matrices, explained in the next remark.

Remark 2.12. Let H be an $(H, 1)$-quasiseparable matrix specified by its generators $\{q_k, d_k, g_k, b_k, h_k\}$. Then the matrices $H(0, 1, 0, 1, 0, \ldots)$ and $H(1, 0, 1, 0, 1, \ldots)$ are five-diagonal. In particular,

$$H(0, 1, 0, 1, 0, \ldots) = \begin{bmatrix} d_1 & g_1 & 0 & & & & & \\ q_1 h_2 & d_2 & q_2 h_3 & q_2 b_3 & & & & \\ q_1 b_2 & g_2 & d_3 & g_3 & 0 & & & \\ & 0 & q_3 h_4 & d_4 & q_4 h_5 & q_4 b_5 & & \\ & & q_3 b_4 & g_4 & d_5 & g_5 & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

and $H(1, 0, 1, 0, 1, \ldots) = H(0, 1, 0, 1, 0, \ldots)^T$. Thus for every $(H, 1)$-quasiseparable matrix there always exists at least one five-diagonal twisted $(H, 1)$-quasiseparable matrix having the same system of characteristic polynomials. More details on five-diagonal matrices will be given in Section 4.

3. A survey of [20] results for $(H, 1)$-quasiseparable polynomials

In the present section we briefly describe the main results of [20] in order to use them extensively in Section 4.

3.1. A bijection between strictly Hessenberg matrices and polynomial systems

Let $\mathbb{H}_n$ be the set of strictly[5] upper Hessenberg $n \times n$ matrices, λ_0 and λ_n be two nonzero parameters, and $\mathbb{P}_n$ be the set of polynomial systems $\{r_k\}_{k=0}^n$ with $\deg r_k = k$. We next demonstrate that there is a bijection between the triple $(\mathbb{H}_n, \lambda_0, \lambda_n)$ and $\mathbb{P}_n$. Indeed, given a polynomial system $\{r_k\}_{k=0}^n$ satisfying $\deg r_k = k$, there exist unique n-term recurrence relations of the form

$$\begin{aligned} & r_0(x) = \tfrac{1}{m_{0,0}}, \quad x \cdot r_{k-1}(x) = m_{k,k} r_k(x) - m_{k-1,k} r_{k-1}(x) - \cdots - m_{0,k} r_0(x), \\ & m_{k,k} \neq 0, \quad k = 1, \ldots, n. \end{aligned} \tag{3.1}$$

This formula represents $x \cdot r_{k-1}$ in the space of all polynomials of degree at most k in terms of $\{r_j\}_{j=0}^k$, which form a basis in that space, and hence these coefficients are unique. Forming a matrix $H \in \mathbb{H}_n$ and parameters λ_0 and λ_n from these coefficients of the form

$$H = \begin{bmatrix} m_{0,1} & m_{0,2} & m_{0,3} & \cdots & m_{0,n} \\ m_{1,1} & m_{1,2} & m_{1,3} & \cdots & m_{1,n} \\ 0 & m_{2,2} & m_{2,3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & m_{n-2,n} \\ 0 & \cdots & 0 & m_{n-1,n-1} & m_{n-1,n} \end{bmatrix}, \quad \lambda_0 = \frac{1}{m_{0,0}}, \quad \lambda_n = \frac{1}{m_{n,n}}, \tag{3.2}$$

it is clear that there is a bijection between $(\mathbb{H}_n, \lambda_0, \lambda_n)$ and $\mathbb{P}_n$, as they share the same unique parameters. Furthermore, it was shown in [26] that the strictly upper Hessenberg matrix H defined in (3.2) and the polynomial system (3.1) are related via (1.1) with $\lambda_k = \frac{1}{m_{k,k}}$. This shows the desired bijection (1.2). The matrix H in (3.2) is usually called *confederate* for the system of polynomials (3.1).

To conclude, for an arbitrary matrix H and scaling factors $\{\lambda_k\}_{k=0}^n$ there exists a unique system of polynomials related to it via (1.1). But the converse is, of course, not true. However, a bijection does exist if we restrict our attention to strictly upper Hessenberg matrices.

3.2. Well-free polynomials and three-term recurrence relations

Consider the general three-term recurrence relations

$$\begin{aligned} & r_0(x) = \alpha_0, \quad r_1(x) = (\alpha_1 x + \beta_1) \cdot r_0(x), \\ & r_k(x) = (\alpha_k x + \beta_k) \cdot r_{k-1}(x) + (\gamma_k x + \delta_k) \cdot r_{k-2}(x), \quad \alpha_k \neq 0. \end{aligned} \tag{3.3}$$

These recurrence relations can be treated as the generalized version of the recurrence relations

$$r_k(x) = (\alpha_k x + \beta_k) r_{k-1}(x) + \gamma_k \cdot r_{k-2}(x), \quad \alpha_k \neq 0, \ \gamma_k > 0 \tag{3.4}$$

satisfied by polynomials orthogonal on the real line, as well as of the recurrence relations (1.10) satisfied by Szegö polynomials (polynomials orthogonal on the unit circle).

Theorem 3.1 (General three-term recurrence relations)**.** *A polynomial system $\{r_k(x)\}_{k=0}^n$ satisfies three-term recurrence relations (3.3) if and only if there exists an $(H, 1)$-quasiseparable matrix H with the set of generators $\{q_k, d_k, g_k, b_k, h_k \neq 0\}$ related to it via (1.1) with $\lambda_k = \frac{1}{q_k}$. Moreover, conversion formulas between generators and recurrence relations coefficients are given in Table 5.*

Remark 3.2. $(H, 1)$-quasiseparable matrices with the restriction $h_k \neq 0$ on the generators were called *well-free* in [20]. Therefore, polynomials satisfying (3.3) are also called *well-free* polynomials.

[5] I.e. having nonzero subdiagonal elements.

Table 5
Conversion formulas: three-term r.r. coefficients $\Longleftrightarrow$ quasiseparable generators.

q_k	d_k	g_k	b_k	h_k
Quasiseparable generators				
$\frac{1}{\alpha_k}$	$-\frac{\alpha_{k-1}\beta_k+\gamma_k}{\alpha_{k-1}\alpha_k}$	$-\frac{\gamma_{k+1}d_k+\delta_k}{\alpha_{k+1}}$	$-\frac{\gamma_{k+1}}{\alpha_{k+1}}$	1
α_k	β_k	γ_k	δ_k	
Three-term r.r. coefficients				
$\frac{1}{q_k}$	$\frac{q_{k-1}b_{k-1}-d_k}{q_k}$	$-\frac{b_{k-1}}{q_k}$	$\frac{d_kb_k-g_k}{q_{k+1}}$	

Table 6
Conversion formulas: Szegö two-term r.r. coefficients $\Longleftrightarrow$ quasiseparable generators.

q_k	d_k	g_k	b_k	h_k
Quasiseparable generators				
$\frac{1}{\delta_k}$	$\theta_k+\frac{\beta_{k-1}\gamma_k}{\delta_{k-1}\delta_k}$	$-\frac{\beta_{k-1}(\alpha_k\delta_k-\beta_k\gamma_k)}{\delta_{k-1}\delta_k}$	$(\alpha_k\delta_k-\beta_k\gamma_k)$	γ_k
α_k	β_k	γ_k	δ_k	θ_k
Szegö-type r.r. coefficients				
$\frac{b_k}{q_k}+\frac{g_{k+1}q_{k+1}}{b_{k+1}}h_k$	$\frac{g_{k+1}q_{k+1}}{b_{k+1}}q_k$	h_k	q_k	$d_k-g_kh_k$

3.3. Semiseparable polynomials and Szegö-type two-term recurrence relations

Another interesting family of recurrence relations for polynomials considered in [20] are the so-called *Szegö-type* two-term recurrence relations, of the form

$$\begin{bmatrix} G_0(x) \\ r_0(x) \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \delta_0 \end{bmatrix}, \quad \begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ (x+\theta_k)\cdot r_{k-1}(x) \end{bmatrix} \tag{3.5}$$

with $\alpha_k\delta_k-\beta_k\gamma_k \neq 0$, $\delta_k \neq 0$ and $G_k(x)$ being auxiliary polynomials.

These recurrence relations generalize those satisfied by Szegö polynomials, of the form

$$\begin{bmatrix} \phi_0(x) \\ \phi_0^{\#}(x) \end{bmatrix} = \frac{1}{\mu_0}\begin{bmatrix} -\rho_0^* \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \phi_k(x) \\ \phi_k^{\#}(x) \end{bmatrix} = \frac{1}{\mu_k}\begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix} \begin{bmatrix} \phi_{k-1}(x) \\ x\phi_{k-1}^{\#}(x) \end{bmatrix}, \tag{3.6}$$

justifying the name Szegö-type.

Theorem 3.3 (Szegö-type two-term recurrence relations)**.** *A polynomial system $\{r_k(x)\}_{k=0}^{n}$ satisfies two-term recurrence relations (3.5) if and only if there exists an $(H,1)$-quasiseparable matrix H with the set of generators $\{q_k, d_k, g_k, b_k \neq 0, h_k\}$ related to it via (1.1) with $\lambda_k = \frac{1}{q_k}$. Moreover, conversion formulas between generators and recurrence relations coefficients are given in Table 6.*

Remark 3.4. $(H,1)$-quasiseparable matrices with the restriction $b_k \neq 0$ on the generators were called *semiseparable* in [20]. Therefore, polynomials satisfying (3.5) are also called *semiseparable* polynomials.

3.4. Quasiseparable polynomials and EGO-type two-term recurrence relations

The authors of [20] established that the class of polynomials related to $(H,1)$-quasiseparable matrices via (1.1) are characterized as those satisfying *EGO-type* two term recurrence relations

$$\begin{bmatrix} F_0(x) \\ r_0(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \theta_0 \end{bmatrix}, \quad \begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \beta_k & \gamma_k \\ \delta_k & \theta_k x+\varepsilon_k \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix} \tag{3.7}$$

with auxiliary polynomials $F_k(x)$.

Table 7
Conversion formulas: *EGO*-type r.r. coefficients $\Longleftrightarrow$ quasiseparable generators.

q_k	d_k	g_k	b_k	h_k
Quasiseparable generators				
$\frac{1}{\theta_k}$	$-\frac{\varepsilon_k}{\theta_k}$	$-\gamma_k$	β_k	$\frac{\delta_k}{\theta_k}$
β_k	γ_k	δ_k	θ_k	ε_k
EGO-type r.r. coefficients				
b_k	$-g_k$	$\frac{h_k}{q_k}$	$\frac{1}{q_k}$	$-\frac{d_k}{q_k}$

Theorem 3.5 (EGO-type two-term recurrence relations)**.** *A polynomial system $\{r_k(x)\}_{k=0}^n$ satisfies two-term recurrence relations (3.7) if and only if there exists an $(H, 1)$-quasiseparable matrix H with the set of generators $\{q_k, d_k, g_k, b_k, h_k\}$ related to it via (1.1) with $\lambda_k = \frac{1}{q_k}$. Moreover, conversion formulas between generators and recurrence relations coefficients are given in Table 7.*

Remark 3.6. Due to the bijection established by Theorem 3.5, it was proposed in [20] to call polynomials satisfying the recurrence relations (3.7) $(H, 1)$-quasiseparable polynomials or, simply, quasiseparable polynomials.

4. Five-diagonal twisted $(H, 1)$-quasiseparable matrices

The results of Section 2 connect five-diagonal CMV matrices (1.4) and Fiedler matrices (1.6) to the theory of quasiseparable matrices (through the concept of twist transformations). In fact, the quasiseparable approach (Section 3) leads to several new results on five-diagonal matrices.

In this section we investigate recurrence relations satisfied by polynomials related to five-diagonal twisted $(H, 1)$-quasiseparable matrices via

$$r_0(x) = \lambda_0, \quad r_k(x) = \lambda_0\lambda_1 \cdots \lambda_k \det(xI - A_{k\times k}), \quad k = 1, \dots, n \tag{4.1}$$

with $\lambda_k \neq 0$ being additional parameters.

We start by deriving an entrywise description of five-diagonal twisted $(H, 1)$-quasiseparable matrices in order to distinguish them from general five-diagonal matrices.

4.1. Full description of five-diagonal twisted $(H, 1)$-quasiseparable matrices

Consider, for example, a 6×6 five-diagonal matrix

$$A = \begin{bmatrix} \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & \star & 0 \\ 0 & \star & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & 0 & 0 & 0 \\ m_{21} & m_{22} & m_{23} & m_{24} & 0 & 0 \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} & 0 \\ 0 & m_{42} & m_{43} & m_{44} & m_{45} & m_{46} \\ 0 & 0 & m_{53} & m_{54} & m_{55} & m_{56} \\ 0 & 0 & 0 & m_{64} & m_{65} & m_{66} \end{bmatrix}. \tag{4.2}$$

This matrix is $(1, 1)$-quasiseparable (Definition 1.1) if and only if all of its submatrices $A(1 : i, i+1, n)$ and $A(i+1 : n, 1 : i)$ for $i = 2, \dots, n-2$ are of rank one. For instance, the submatrix $A(1 : 2, 3 : 6)$ highlighted in (4.2) is of rank one if and only if $m_{13} \cdot m_{24} = 0$. This observation leads to the following simple theorem.

Theorem 4.1 (Characterizations of five-diagonal $(1, 1)$-quasiseparable matrices)**. Entrywise characterization.** *A five-diagonal matrix $A = [m_{ij}]_{i,j=1}^n$ is $(1, 1)$-quasiseparable if and only if*

$$m_{i,i+2} \cdot m_{i+1,i+3} = m_{i+2,i} \cdot m_{i+3,i+1} = 0, \quad i = 1, \dots, n-3. \tag{4.3}$$

Generator characterization. *A* $(1,1)$*-quasiseparable matrix A is five-diagonal if and only if it has a choice of generators such that*

$$a_k \cdot a_{k+1} = b_k \cdot b_{k+1} = 0, \quad k = 2, \ldots, n-2. \tag{4.4}$$

The conditions (4.3) and (4.4) actually imply that in a five-diagonal $(1,1)$-quasiseparable matrix every nonzero entry on the second sub(super)diagonal is surrounded by two zero entries on that sub(super)diagonal.

Presented next are necessary and sufficient conditions for a five-diagonal matrix to be twisted $(H,1)$-quasiseparable.

Theorem 4.2. *A five-diagonal matrix* $A = [m_{ij}]_{i,j=1}^n$ *is twisted* $(H,1)$*-quasiseparable if and only if in addition to* (4.3) *it also satisfies*

$$m_{i,i+2} \cdot m_{i+2,i} = 0, \quad i = 1, \ldots, n-2. \tag{4.5}$$

Proof. We first prove the "only if" implication. Let $A = [m_{ij}]$ be a five-diagonal twisted $(H,1)$-quasiseparable matrix. Since A is also $(1,1)$-quasiseparable it satisfies the conditions of Theorem 4.1 and, hence

$$m_{i,i+2} \cdot m_{i+1,i+3} = m_{i+2,i} \cdot m_{i+3,i+1} = 0, \quad i = 1, \ldots, n-2. \tag{4.6}$$

Let A be described by a set of $(1,1)$-quasiseparable generators as in Definition 1.2. Then its generators satisfy (see Definition 2.8) $a_i \cdot b_i = 0$ which implies

$$m_{i,i+2} \cdot m_{i+2,i} = 0, \quad i = 1, \ldots, n-2.$$

Therefore, the "only if" implication is proved.

Next, let $A = [m_{ij}]$ be a five-diagonal matrix satisfying conditions (4.3) and (4.5). These conditions imply that A is $(1,1)$-quasiseparable (see Theorem 4.1). Hence, A has a set of generators as in Definition 1.2. Because of the condition (4.6) and five-diagonality we can always choose generators to be

$$a_i = 0 \quad \text{if } m_{i+2,1} = 0 \quad \text{and} \quad b_i = 0 \quad \text{if } m_{i,i+2} = 0.$$

Applying a twist transformation (Definition 2.1) for corresponding indices we can convert matrix A. $\square$

4.2. Non-uniqueness of five-diagonal twisted $(H,1)$-quasiseparable matrices

As we have seen in Section 3.1 there is a bijection between $(H,1)$-quasiseparable matrices together with two nonzero parameters q_0 and q_n and systems of polynomials related to them via (4.1) with $\lambda_k = \frac{1}{q_k}$, where q_k, $k = 1, \ldots, n-1$ are generators as in Definition 2.5. In contrast to this bijection for a given system of polynomials (related to some $(H,1)$-quasiseparable matrix) there are infinitely many five-diagonal twisted $(H,1)$-quasiseparable matrices related to it via (4.1). Next, we describe two reasons for this non-uniqueness.

Reason 1 (*Non-uniqueness of patterns*)**.** Let H be an $(H,1)$-quasiseparable matrix with generators $\{q_k, d_k, g_k, b_k, h_k\}$. Then the twisted $(H,1)$-quasiseparable matrices with patterns $(\bigstar, 1, 0, 1, 0, \ldots)$ and $(\bigstar, 0, 1, 0, 1, \ldots)$ obtained from H via twist transformations are five-diagonal. For example,

$$H(0,1,0,1,0,\ldots) = \begin{bmatrix} d_1 & g_1 & 0 & & & & \\ q_1 h_2 & d_2 & q_2 h_3 & q_2 b_3 & & & \\ q_1 b_2 & g_2 & d_3 & g_3 & 0 & & \\ & 0 & q_3 h_4 & d_4 & q_4 h_5 & q_4 b_5 & \\ & & q_3 b_4 & g_4 & d_5 & g_5 & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{4.7}$$

and $H(1, 0, 1, 0, 1, \ldots) = H(0, 1, 0, 1, 0, \ldots)^T$. Moreover, all five-diagonal twisted $(H, 1)$-quasiseparable matrices obtained from H share the same system of polynomials (Theorem 2.2).

Remark 4.3. Five-diagonal matrices of the following zero patterns

$$\begin{bmatrix} \star & \star & 0 & & & & & \\ \star & \star & \star & \star & & & & \\ \star & \star & \star & \star & 0 & & & \\ & 0 & \star & \star & \star & \star & & \\ & & \star & \star & \star & \star & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \quad \begin{bmatrix} \star & \star & \star & & & & & \\ \star & \star & \star & 0 & & & & \\ 0 & \star & \star & \star & \star & & & \\ & \star & \star & \star & \star & 0 & & \\ & & 0 & \star & \star & \star & \star & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{4.8}$$

always satisfy restrictions (4.3) and (4.5) and, hence, are always twisted $(H, 1)$-quasiseparable. In addition, any matrices of patterns $(\star, 1, 0, 1, 0, \ldots)$ and $(\star, 0, 1, 0, 1, \ldots)$ have the first and second zero patterns of (4.8), respectively.

Another fact which is worth mentioning is that if an $(H, 1)$-quasiseparable matrix H has generators such that $b_k = 0$ for some k, then there are two (possibly distinct) five-diagonal twisted $(H, 1)$-quasiseparable matrices of patterns

$$H(i_1, \ldots, i_{k-1}, 0, i_{k+1}, \ldots, i_n) \quad \text{and} \quad H(i_1, \ldots, i_{k-1}, 1, i_{k+1}, \ldots, i_n).$$

Reason 2 (*Non-uniqueness of quasiseparable generators*). Let $H(0, 0, 0, 0)$ be a 4×4 $(H, 1)$-quasiseparable matrix specified by its generators and such that its $H(1 : 2, 3 : 4)$ block is zero (all other entries are not zeros)

$$H(0, 0, 0, 0) = \begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & g_1b_2b_3h_4 \\ q_1 & d_2 & g_2h_3 & g_2b_3h_4 \\ 0 & q_2 & d_3 & g_3h_4 \\ 0 & 0 & q_3 & d_4 \end{bmatrix} = \begin{bmatrix} d_1 & g_1h_2 & 0 & 0 \\ q_1 & d_2 & 0 & 0 \\ 0 & q_2 & d_3 & g_3h_4 \\ 0 & 0 & q_3 & d_4 \end{bmatrix}$$

Clearly, such a matrix can be described by two different sets of generators:

$(\mathbf{A}) g_2 = b_2 = 0, \quad b_3, h_3$ —arbitrary,

or

$(\mathbf{B}) b_3 = h_3 = 0, \quad g_2, b_2$ —arbitrary.

Performing an elementary twist transformations for indices 2 and 4 we obtain a five-diagonal analog of matrix $H(0, 0, 0, 0)$:

$$H(0, 1, 0, 1) = \begin{bmatrix} d_1 & g_1 & 0 & 0 \\ h_2q_1 & d_2 & q_2h_3 & q_2b_3h_4 \\ b_2q_1 & g_2 & d_3 & g_3 \\ 0 & 0 & h_4q_3 & d_4 \end{bmatrix}.$$

Different choices of generators lead to *significantly* different five-diagonal matrices:

$$(\mathbf{A}) \begin{bmatrix} d_1 & g_1 & 0 & 0 \\ h_2q_1 & d_2 & q_2h_3 & q_2b_3h_4 \\ 0 & 0 & d_3 & g_3 \\ 0 & 0 & h_4q_3 & d_4 \end{bmatrix}, \quad (\mathbf{B}) \begin{bmatrix} d_1 & g_1 & 0 & 0 \\ h_2q_1 & d_2 & 0 & 0 \\ b_2q_1 & g_2 & d_3 & g_3 \\ 0 & 0 & h_4q_3 & d_4 \end{bmatrix}.$$

4.3. General three-term recurrence relations

Let us recall that under the additional restriction of $\rho_k \neq 0$ for each k, the corresponding Szegö polynomials satisfy three-term recurrence relations

Table 8
Conversion formulas for the generators of a twisted $(H, 1)$-quasiseparable matrix $A(i_1 \dots i_n)$ in terms of the corresponding three-term recurrence relation coefficients $\{\alpha_k, \beta_k, \gamma_k, \delta_k\}$.

	g_k	b_k	h_k	d_k	p_k	a_k	q_k
If $i_k = 0$	$-\frac{\gamma_{k+1}d_k+\delta_k}{\alpha_{k+1}}$	$-\frac{\gamma_{k+1}}{\alpha_{k+1}}$	1	$-\frac{\alpha_{k-1}\beta_k+\gamma_k}{\alpha_{k-1}\alpha_k}$	1	$-\frac{\gamma_{k+1}}{\alpha_{k+1}}$	$-\frac{\gamma_{k+1}d_k+\delta_k}{\alpha_{k+1}}$
If $i_k = 1$	$\frac{1}{\alpha_k}$	0	1	$-\frac{\alpha_{k-1}\beta_k+\gamma_k}{\alpha_{k-1}\alpha_k}$	1	0	$\frac{1}{\alpha_k}$

$$\begin{aligned} &\phi_0^\#(x) = \frac{1}{\mu_0}, \quad \phi_1^\#(x) = \frac{1}{\mu_1}(x \cdot \phi_0^\#(x) + \rho_1\rho_0^* \cdot \phi_0^\#(x)), \\ &\phi_k^\#(x) = \left[\frac{1}{\mu_k} \cdot x + \frac{\rho_k}{\rho_{k-1}}\frac{1}{\mu_k}\right]\phi_{k-1}^\#(x) - \frac{\rho_k}{\rho_{k-1}}\frac{\mu_{k-1}}{\mu_k} \cdot x \cdot \phi_{k-2}^\#(x), \quad k = 2, \dots, n. \end{aligned} \tag{4.9}$$

This system of polynomials corresponds to a five-diagonal twisted $(H, 1)$-quasiseparable matrix (in fact, the CMV matrix). The theorem below generalizes this observation giving necessary and sufficient conditions for the existence of general three-term recurrence relations for polynomials in terms of five-diagonal matrices they are related to via (4.1).

Theorem 4.4. *A system of polynomials $R = \{r_k(x)\}_{k=0}^n$ satisfies three-term recurrence relations*

$$\begin{aligned} &r_0(x) = \alpha_0, \quad r_1(x) = (\alpha_1 x + \beta_1) \cdot r_0(x), \\ &r_k(x) = (\alpha_k x + \beta_k) \cdot r_{k-1}(x) + (\gamma_k x + \delta_k) \cdot r_{k-2}(x), \quad \alpha_k \neq 0. \end{aligned} \tag{4.10}$$

if and only if it is related via (4.1) to a five-diagonal matrix A having zero pattern

$$\begin{bmatrix} \star & \star & 0 & & & & \\ \boxed{\star} & \star & \boxed{\star} & \star & & & \\ \star & \star & \star & \star & 0 & & \\ & 0 & \boxed{\star} & \star & \boxed{\star} & \star & \\ & & \star & \star & \star & \star & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{4.11}$$

with nonzero highlighted entries via (4.1) with $\lambda_k = \alpha_k$.

Proof. [**Necessity**] Obviously, the matrix A is twisted $(H, 1)$-quasiseparable (see Remark 4.3) and its general representation is

$$A = \begin{bmatrix} d_1 & g_1 & 0 & & & & \\ \boxed{q_1h_2} & d_2 & \boxed{q_2h_3} & q_2b_3 & & & \\ q_1b_2 & g_2 & d_3 & g_3 & 0 & & \\ & 0 & \boxed{q_3h_4} & d_4 & \boxed{q_4h_5} & q_4b_5 & \\ & & q_3b_4 & g_4 & d_5 & g_5 & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{4.12}$$

Letting $q_k = \frac{1}{\lambda_k}$, then all other generators $\{d_k, g_k, b_k, h_k\}$ are defined uniquely. Moreover, the generators h_k are all nonzero. Hence, there exists a unique $(H, 1)$-quasiseparable matrix H having generators $\{q_k, d_k, g_k, b_k, h_k \neq 0\}$ and according to Theorem 2.2 it is related to the system of polynomials R via (4.1). It was proved in Theorem 3.1 that polynomials related to $(H, 1)$-quasiseparable matrices with $h_k \neq 0$ satisfy the recurrence relations (4.12) and, hence, so do $\{r_k(x)\}_{k=0}^n$.

[**Sufficiency**] Let R satisfy three-term recurrence relations (4.10), it follows from Theorem 3.1 that there exists a unique $(H, 1)$-quasiseparable matrix H with $q_k = \frac{1}{\alpha_k}$ and $h_k \neq 0$ such that it is related to R via (4.1) with $\lambda_k = \alpha_k$. Let $A = H(0, 1, 0, 1, \dots)$ be a five-diagonal twisted $(H, 1)$-quasiseparable matrix obtained from H via twist transformations, then it has the zero pattern (4.11) and by Theorem 2.2 is related to the system of polynomials R via (4.1) with $\lambda_k = \alpha_k$. $\square$

The following corollary follows directly from Theorem 3.1, Remark 4.3, and Theorem 4.4.

Corollary 4.5. *A system of polynomials satisfies three-term recurrence relations (4.10) if and only if it is related via (4.1) to a twisted $(H, 1)$-quasiseparable matrix A with generators $\{q_k, d_k, g_k, b_k, h_k \neq 0\}$.*[6] *Table 8 gives a conversion from three-term recurrence relation coefficients to the generators of the matrix A.*

Example 4.6. From the CMV matrix

$$\mathcal{K} = \begin{bmatrix} -\rho_0^*\rho_1 & \rho_0^*\mu_1 & 0 & & & & \\ -\mu_1\rho_2 & -\rho_1^*\rho_2 & -\mu_2\rho_3 & \mu_2\mu_3 & & & \\ \mu_1\mu_2 & \rho_1^*\mu_2 & -\rho_2^*\rho_3 & \rho_2^*\mu_3 & 0 & & \\ & 0 & -\mu_3\rho_4 & -\rho_3^*\rho_4 & -\mu_4\rho_5 & \mu_4\mu_5 & \\ & & \mu_3\mu_4 & \rho_3^*\mu_4 & -\rho_4^*\rho_5 & \rho_4^*\mu_5 & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{4.13}$$

it is easy to see that the additional restriction $\rho_k \neq 0$ implies that all highlighted entries in (4.13) are not zeros[7] and, hence, the matrix $\mathcal{K}$ satisfies conditions of Theorem 4.4. This proves the existence of recurrence relations (4.9) for polynomials related to $\mathcal{K}$ via (4.1).

Example 4.7. Applying Theorem 4.4 to the transpose of a Fiedler matrix

$$F^T = \begin{bmatrix} -m_1 & 1 & 0 & & & & & \\ -m_2 & 0 & -m_3 & 1 & & & & \\ 1 & 0 & 0 & 0 & 0 & & & \\ & 0 & -m_4 & 0 & -m_5 & 1 & & \\ & & 1 & 0 & 0 & 0 & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{4.14}$$

we conclude that three-term recurrence relations (4.10) must exist for Horner polynomials under the condition $m_k \neq 0$ for $k = 2, \ldots, n$. Indeed, from the recurrence relations $p_k(x) = x \cdot p_{k-1}(x) + m_k$ we can get that

$$1 = \frac{p_{k-1} - x \cdot p_{k-2}}{m_{k-1}},$$

and, hence

$$p_k(x) = x \cdot p_{k-1}(x) + m_k \cdot 1 = \left(x + \frac{m_k}{m_{k-1}}\right) p_{k-1}(x) - \frac{m_k}{m_{k-1}} x \cdot p_{k-2}.$$

4.4. Szegö-type two-term recurrence relations

It is well-known that Szegö polynomials related to CMV matrices via (4.1) satisfy two-term recurrence relations

$$\begin{bmatrix} \phi_0(x) \\ \phi_0^{\#}(x) \end{bmatrix} = \frac{1}{\mu_0} \begin{bmatrix} -\rho_0^* \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \phi_k(x) \\ \phi_k^{\#}(x) \end{bmatrix} = \frac{1}{\mu_k} \begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix} \begin{bmatrix} \phi_{k-1}(x) \\ x\phi_{k-1}^{\#}(x) \end{bmatrix}. \tag{4.15}$$

In this section we consider the general form of recurrence relations (4.15) (which were called Szegö-type in [20]) and derive the class of five-diagonal twisted $(H, 1)$-quasiseparable matrices related via (4.1) to polynomials satisfying them.

[6] Such matrices are called twisted $(H, 1)$-well-free, see Remark 3.2.

[7] The definition of the complementary parameters μ_k is $\mu_k = \begin{cases} \sqrt{1 - |\rho_k|^2} & |\rho_k| < 1, \\ 1 & |\rho_k| = 1, \end{cases}$ which are thus always nonzero.

Table 9
Conversion formulas for the generators of a twisted $(H, 1)$-quasiseparable matrix $A(i_1 \dots i_n)$ in terms of the corresponding Szegö-type recurrence relation coefficients $\{\alpha_k, \beta_k, \gamma_k, \delta_k, \theta_k\}$.

	g_k	b_k	h_k	d_k	p_k	a_k	q_k
If $i_k = 0$	$-\frac{\beta_{k-1}(\alpha_k\delta_k-\beta_k\gamma_k)}{\delta_{k-1}\delta_k}$	$(\alpha_k\delta_k - \beta_k\gamma_k)$	γ_k	$\theta_k + \frac{\beta_{k-1}\gamma_k}{\delta_{k-1}\delta_k}$	1	0	$\frac{1}{\delta_k}$
If $i_k = 1$	$\frac{1}{\delta_k}$	0	1	$\theta_k + \frac{\beta_{k-1}\gamma_k}{\delta_{k-1}\delta_k}$	γ_k	$(\alpha_k\delta_k - \beta_k\gamma_k)$	$-\frac{\beta_{k-1}(\alpha_k\delta_k-\beta_k\gamma_k)}{\delta_{k-1}\delta_k}$

Theorem 4.8. *A system of polynomials $R = \{r_k(x)\}_{k=0}^n$ satisfies Szegö-type two-term recurrence relations*

$$\begin{bmatrix} G_0(x) \\ r_0(x) \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \delta_0 \end{bmatrix}, \quad \begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ (x+\theta_k)\cdot r_{k-1}(x) \end{bmatrix} \tag{4.16}$$

with $\alpha_k\delta_k - \beta_k\gamma_k \neq 0$, $\delta_k \neq 0$ if and only if it is related via (4.1) to a five-diagonal matrix A having zero pattern

$$\begin{bmatrix} \star & \star & 0 & & & & \\ \star & \star & \star & \boxed{\star} & & & \\ \boxed{\star} & \star & \star & \star & 0 & & \\ & 0 & \star & \star & \star & \boxed{\star} & \\ & & \boxed{\star} & \star & \star & \star & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{4.17}$$

with nonzero highlighted entries via (4.1) with $\lambda_k = \delta_k$.

Proof. [**Necessity**] The matrix A is twisted $(H, 1)$-quasiseparable (see Remark 4.3) and its general representation is

$$A = \begin{bmatrix} d_1 & g_1 & 0 & & & & \\ q_1h_2 & d_2 & q_2h_3 & \boxed{q_2b_3} & & & \\ \boxed{q_1b_2} & g_2 & d_3 & g_3 & 0 & & \\ & 0 & q_3h_4 & d_4 & q_4h_5 & \boxed{q_4b_5} & \\ & & \boxed{q_3b_4} & g_4 & d_5 & g_5 & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{4.18}$$

Fixing $q_k = \frac{1}{\lambda_k}$, then all other generators $\{d_k, g_k, b_k, h_k\}$ are defined uniquely. Moreover, the generators b_k are all nonzero. Hence, there exists a unique $(H, 1)$-quasiseparable matrix H having generators $\{q_k, d_k, g_k, b_k \neq 0, h_k\}$ and according to Theorem 2.2 it is related to the system of polynomials R via (4.1). It is proved in Theorem 3.3 that polynomials related to $(H, 1)$-quasiseparable matrices with $b_k \neq 0$ satisfy recurrence relations (4.16) and, hence, so do $\{r_k(x)\}_{k=0}^n$.

[**Sufficiency**] Let R satisfy Szegö-type recurrence relations (4.16). Then Theorem 3.3 implies that there exists a unique $(H, 1)$-quasiseparable matrix H with $q_k = \frac{1}{\delta_k}$ and $b_k \neq 0$ that is related to R via (4.1) with $\lambda_k = \delta_k$. Let $A = H(0, 1, 0, 1, \dots)$ be a five-diagonal twisted $(H, 1)$-quasiseparable matrix obtained from H via twist transformations, which has the zero pattern (4.17) and by Theorem 2.2 is related to the system of polynomials R via (4.1) with $\lambda_k = \delta_k$. □

The following corollary follows directly from Theorem 3.3, Remark 4.3, and Theorem 4.8.

Corollary 4.9. *A system of polynomials satisfies Szegö-type two-term recurrence relations (4.16) if and only if it is related via (4.1) to a twisted $(H, 1)$-quasiseparable matrix A with generators $\{q_k, d_k, g_k, b_k \neq 0, h_k\}$.*[8] *Table 9 gives a conversion from Szegö-type recurrence relation coefficients to the generators of the matrix A.*

[8] Such matrices are called twisted $(H, 1)$-semiseparable, see Remark 3.4.

Example 4.10. Let us note also that the transpose of the Fiedler matrix (1.6) satisfies the conditions of Theorem 4.8,

$$F^T = \begin{bmatrix} -m_1 & 1 & 0 & & & & & \\ -m_2 & 0 & -m_3 & \boxed{1} & & & & \\ \boxed{1} & 0 & 0 & 0 & 0 & & & \\ & 0 & -m_4 & 0 & -m_5 & \boxed{1} & & \\ & & \boxed{1} & 0 & 0 & 0 & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{4.19}$$

as the highlighted entries in (4.19) are nonzeros. Hence, Horner polynomials satisfy Szegö-type recurrence relations (4.15). Using the generators from Table 3 and the conversion formulas listed in Table 6, we arrive at the following Szegö-type recurrence relations for Horner polynomials,

$$\begin{bmatrix} F_0(x) \\ p_0(x) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} F_k(x) \\ p_k(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ m_k & 1 \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ x \cdot p_{k-1}(x) \end{bmatrix}. \tag{4.20}$$

4.5. EGO-type two-term recurrence relations

It was observed in Section 4.1 that five-diagonal twisted $(H, 1)$-quasiseparable matrices form a proper subclass of $(1, 1)$-quasiseparable matrices. Hence, it is natural to expect that polynomials related to them via (4.1) satisfy some special recurrence relations rather than the general recurrence relations (1.7). The next theorem shows that this is, indeed, the case.

Theorem 4.11. *A system of polynomials $R = \{r_k(x)\}_{k=0}^n$ satisfies EGO-type two-term recurrence relations*

$$\begin{bmatrix} F_0(x) \\ r_0(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \theta_0 \end{bmatrix}, \quad \begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \beta_k & \gamma_k \\ \delta_k & \theta_k x + \varepsilon_k \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix} \tag{4.21}$$

with $\theta_k = \lambda_k$ if and only if it is related via (4.1) to a five-diagonal matrix $A = [m_{ij}]_{ij=1}^n$ with entries satisfying

$$\begin{aligned} &m_{i,i+2} \cdot m_{i+1,i+3} = m_{i+2,i} \cdot m_{i+3,i+1} = 0, \quad i = 1, \ldots, n-3, \\ &m_{i,i+2} \cdot m_{i+2,1} = 0, \quad i = 1, \ldots, n-2. \end{aligned} \tag{4.22}$$

Proof. [**Necessity**] Let the entries of A satisfy (4.22). Then according to Theorem 4.2, A is twisted $(H, 1)$-quasiseparable. Hence, there exists an $(H, 1)$-quasiseparable matrix related via (4.1) to the same system of polynomials as A by Theorem 2.2. It immediately follows from Theorem 3.5 that the related polynomials R satisfy the recurrence relations (4.21).

[**Sufficiency**] Let R satisfy EGO-type recurrence relations (4.21). Then by Theorem 3.5, there exists an $(H, 1)$-quasiseparable matrix H that is related to R via (4.1). Let $A = H(0, 1, 0, 1, \ldots)$ be a twisted $(H, 1)$-quasiseparable matrix obtained from H via twist transformations. Then it is five-diagonal (see Remark 4.3), satisfies the conditions (4.22) and is related to the system of polynomials R via (4.1) as desired. □

The following corollary follows directly from Theorems 3.5 and 4.11.

Corollary 4.12. *A system of polynomials satisfies EGO-type recurrence relations (4.21) if and only if it is related via (4.1) to a twisted $(H, 1)$-quasiseparable matrix A. Table 10 gives a conversion from EGO-type recurrence relation coefficients to the generators of the matrix A.*

Let us recall that both Fiedler matrices (1.6) and CMV matrices (1.4) fulfill the conditions of Theorem 4.11. Hence, Horner and Szegö polynomials must satisfy EGO-type recurrence relations (4.21). In particular, one can easily check that Horner polynomials satisfy

Table 10
Conversion formulas for the generators of a twisted $(H,1)$-quasiseparable matrix $A(i_1,\dots,i_n)$ in terms of the corresponding EGO-type recurrence relation coefficients $\{\beta_k,\gamma_k,\delta_k,\theta_k,\varepsilon_k\}$.

	g_k	b_k	h_k	d_k	p_k	a_k	q_k
If $i_k=0$	$-\gamma_k$	β_k	$\frac{\delta_k}{\theta_k}$	$-\frac{\varepsilon_k}{\theta_k}$	1	0	$\frac{1}{\theta_k}$
If $i_k=1$	$\frac{1}{\theta_k}$	0	1	$-\frac{\varepsilon_k}{\theta_k}$	$\frac{\delta_k}{\theta_k}$	β_k	$-\gamma_k$

$$\begin{bmatrix}F_0(x)\\p_0(x)\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix},\quad\begin{bmatrix}F_1(x)\\p_1(x)\end{bmatrix}=\begin{bmatrix}1\\x+m_1\end{bmatrix},\quad\begin{bmatrix}F_k(x)\\p_k(x)\end{bmatrix}=\begin{bmatrix}1&0\\m_k&x\end{bmatrix}\begin{bmatrix}F_{k-1}(x)\\p_{k-1}(x)\end{bmatrix}. \tag{4.23}$$

Similarly, Szegö polynomials satisfy

$$\begin{bmatrix}F_0(x)\\\phi_0^{\#}(x)\end{bmatrix}=\begin{bmatrix}0\\\frac{1}{\mu_0}\end{bmatrix},\quad\begin{bmatrix}F_k(x)\\\phi_k^{\#}(x)\end{bmatrix}=\begin{bmatrix}\mu_k&\rho_{k-1}^*\mu_k\\\frac{\rho_k}{\mu_k}&\frac{1}{\mu_k}x-\frac{\rho_{k-1}^*\rho_k}{\mu_k}\end{bmatrix}\begin{bmatrix}F_{k-1}(x)\\\phi_{k-1}^{\#}(x)\end{bmatrix}. \tag{4.24}$$

Interested reader can find more on the connection between five-diagonal matrices and polynomial systems in the sequel paper [24].

5. Nested factorization of twisted $(H,1)$-quasiseparable matrices

In this section we derive a nested factorization of twisted $(H,1)$-quasiseparable matrices which we believe might be useful in developing fast algorithms. The interpretation of the twist transformation (Definition 2.1) can be also given in terms of this new factorization.

Theorem 5.1. *Let H be an arbitrary $(H,1)$-quasiseparable matrix specified by its generators as in Definition 2.5, and define the matrices Θ_k, Δ_k by*

$$\Theta_1=\left[\begin{array}{cc|c}d_1&g_1&\\q_1&d_2&\\\hline&&I_{n-2}\end{array}\right],\quad\Delta_1=O_n,$$
$$\Theta_k=\left[\begin{array}{c|cc|c}I_{k-1}&&&\\\hline&h_k&b_k&\\&q_k&d_{k+1}&\\\hline&&&I_{n-k-1}\end{array}\right],$$
$$\Delta_k=\left[\begin{array}{c|cc|c}O_{k-1}&&&\\\hline&d_k-d_kh_k&g_k-d_kb_k&\\&0&0&\\\hline&&&O_{n-k-1}\end{array}\right], \tag{5.1}$$
$$\text{for}\quad k=2,\dots,n-1,$$
$$\Theta_n=\left[\begin{array}{c|c}I_{n-1}&\\\hline&h_n\end{array}\right],\quad\Delta_n=\left[\begin{array}{c|c}O_{n-1}&\\\hline&d_n-d_nh_n\end{array}\right],$$

where O_k denotes the $k\times k$ zero matrix. Then the decomposition

$$H=(\cdots((\Theta_1\Theta_2+\Delta_2)\Theta_3+\Delta_3)\cdots)\Theta_n+\Delta_n \tag{5.2}$$

holds.

The proof of this decomposition is deferred; it will be seen as a special case of Theorem 5.2. Eq. (5.2) of this theorem can be viewed as forming the $(H,1)$-quasiseparable matrix H by the iteration

$$H_0=I_n,\quad H_k=H_{k-1}\Theta_k+\Delta_k,\quad k=1,\dots,n,\quad H=H_n. \tag{5.3}$$

The next theorem gives the concept of a twist transformation in terms of this decomposition. It states that the effect of an elementary twist transformation at index k changes step k of the decomposition (5.3) via a transpose-like operation to

$$H_k = \Delta_k^T + \Theta_k^T H_{k-1};$$

that is,

$$\begin{array}{c} \hline H_k = H_{k-1}\Theta_k + \Delta_k \\ \updownarrow \\ H_k = \Delta_k^T + \Theta_k^T H_{k-1} \\ \hline \text{Twist transformation in} \\ \text{terms of decomposition} \\ \hline \end{array} \quad \Longleftrightarrow \quad \begin{array}{ccc} \hline p_k & a_k & q_k \\ \updownarrow & \updownarrow & \updownarrow \\ h_k & b_k & g_k \\ \hline \multicolumn{3}{l}{\text{Twist transformation in}} \\ \multicolumn{3}{l}{\text{terms of generators}} \\ \hline \end{array}$$

Theorem 5.2. *Let H be a twisted $(H, 1)$-quasiseparable matrix of pattern $(i_1, i_2, \ldots, i_n)$ with generators $\{q_k, d_k, g_k, b_k, h_k\}$. Then it can be constructed by the following procedure*:

$$H_0 = I_n, \quad H_k = \begin{cases} H_{k-1}\Theta_k + \Delta_k & \text{if } i_k = 0, \\ \Delta_k^T + \Theta_k^T H_{k-1} & \text{if } i_k = 1, \end{cases} \quad k = 1, \ldots n, \quad H = H_n. \tag{5.4}$$

The proof of this theorem is given next, and we note briefly that Theorem 5.1 follows as a corollary with $i_k = 0$ for $k = 1, \ldots, n$.

Proof. We will show by induction that for every $k = 2, \ldots, n$:

$$H_{k-1}(1 : k, 1 : k) = H(1 : k, 1 : k)|_{p_k = h_k = 1}. \tag{5.5}$$

In other words, every kth principal submatrix of H_{k-1} almost equals to the corresponding one of H and equals identically if we modify the last generators p_k and h_k.

The basis of induction ($k = 2$) is trivial:

$$H(2 : 2, 2 : 2) = \begin{bmatrix} d_1 & g_1h_2 \\ p_2q_1 & d_2 \end{bmatrix}, \quad H_1(2 : 2, 2 : 2) = \begin{bmatrix} d_1 & g_1 \\ q_1 & d_2 \end{bmatrix}, \quad \text{for } i_1 = 0,$$
$$H(2 : 2, 2 : 2) = \begin{bmatrix} d_1 & q_1h_2 \\ p_2g_1 & d_2 \end{bmatrix}, \quad H_1(2 : 2, 2 : 2) = \begin{bmatrix} d_1 & q_1 \\ g_1 & d_2 \end{bmatrix}, \quad \text{for } i_1 = 1.$$

Assume that (5.5) holds for all indices up to k. Consider case $i_k = 0$, the remaining case $i_k = 1$ is essentially the same.

Consider matrix $H_k(1 : k + 1, 1 : k + 1)$:

$$\left[\begin{array}{ccc|c} & & & 0 \\ & H_{k-1}(1 : k, 1 : k) & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right] \begin{bmatrix} \ddots & & & \\ & 1 & & \\ & & h_k & b_k \\ & & q_k & d_{k+1} \end{bmatrix} + \begin{bmatrix} \ddots & & & \\ & 0 & & \\ & & d_k - d_kh_k & g_k - d_kb_k \\ & & 0 & 0 \end{bmatrix}. \tag{5.6}$$

The last kth column of matrix $H_{k-1}(1 : k, 1 : k)$ equals the last column of matrix $H(1 : k, 1 : k)$ if in the last we set $h_k = 1$. Therefore, performing the matrix product in (5.6) we get:

$$\left[\begin{array}{ccc|c|c} & H_{k-1}(1 : k, 1 : k - 1) & & \begin{array}{c} H_{k-1}(1 : k - 1, k)h_k \\ d_kh_k + (d_k - d_kh_k) \end{array} & \begin{array}{c} H_{k-1}(1 : k - 1, k)b_k \\ d_kb_k + (g_k - d_kb_k) \end{array} \\ \hline 0 & \cdots & 0 & q_k & d_{k+1} \end{array}\right], \tag{5.7}$$

which is equal to $H(1 : k + 1, 1 : k + 1)|_{p_{k+1} = h_{k+1} = 1}$.

By the induction we get that

$$H_{n-1} = H|_{p_n=h_n=1}.$$

Substituting this into recursions (5.4) we get

$$H_n = H|_{p_n=h_n=1} \begin{bmatrix} I_{n-1} & \\ & h_n \end{bmatrix} + \begin{bmatrix} O_{n-1} & \\ & d_n - d_n h_n \end{bmatrix} = H, \quad \text{if } i_n = 0,$$
$$H_n = \begin{bmatrix} I_{n-1} & \\ & h_n \end{bmatrix} H|_{p_n=h_n=1} + \begin{bmatrix} O_{n-1} & \\ & d_n - d_n h_n \end{bmatrix} = H, \quad \text{if } i_n = 1. \qquad \square$$

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